THE SPHERE IN THE IMAGE

BY

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Introduction. If $f(x)$ is an analytic and univalent function of the complex variable x in the unit circle $S = \{x: |x| < 1\}$ and $f(0) = y_0$, then the image $f(S)$ of S contains a circle with radius $r_0 = \frac{1}{4}|f'(0)|$. Even when $f(x)$ is not univalent, the image $f(S)$ of S contains a circle with radius $k|f'(0)|$, where k is the constant of Bloch(1).

The aim of this paper is to prove theorems, similar to the last statement, for some mappings of n-dimensional Euclidean spaces and general Banach spaces into themselves. The idea of the proofs consists in the following use of fixed point theorems(2):

Suppose that $S = \{x: \|x\| \leqq 1\}$ is the unit sphere in a Banach space X and that for a given $y \in X$ the mapping $x - \beta[f(x) - y]$ of S into X has a fixed point $x \in S$ for some $\beta \neq 0$. Then $f(x) = y$ and so y is contained in the image $f(S)$ of S. Considering all such points y, we are looking for conditions under which the set of these points contains a sphere with radius as large as possible.

By $S = S(x_0, r) = \{x: \rho(x_0, x) \leqq r\}$ we denote the sphere with center x_0 and radius r in a metric space with metric ρ and by $\mathrm{Bd}(S) = \{x: \rho(x_0, x) = r\}$ the boundary of S. (By (x, y) we denote the scalar product of x and y.

1. In this section a simple generalization to Hilbert spaces of the fixed-point theorem of Schauder (Theorem 1) and its applications are given. Before proceeding with the proof of Theorem 1, let us first note the following

LEMMA 1. *If X_1 and X_2 are closed subsets of a metric space X and $f_1: X_1 \to Y$ and $f_2: X_2 \to Y$ are mappings(3) of X_1 and X_2 into a metric space Y, then*

$$f(x) = \begin{cases} f_1(x) & \text{for } x \in X_1 \\ f_2(x) & \text{for } x \in X_2 \end{cases}$$

is continuous on $X_1 \cup X_2$, provided that $f_1(x) = f_2(x)$ on $X_1 \cap X_2$.
The proof is trivial.

REMARK 1. Easy examples show that the assumption of closedness of the sets X_1 and X_2 in Lemma 1 is essential.

(1) Bloch's Theorem has been generalized to mappings of n-dimensional spaces by S. Bochner in [2].

(2) A particular case of this idea has been used in [9], p. 734.

(3) By "mapping" we always understand a continuous mapping.

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THEOREM 1. *Let $f: S \to X$ be a completely continuous mapping of the sphere $S = \{x: \| x - x_0 \| \leqq r\}$ in the Hilbert space X into X, such that*

$$(x - x_0, f(x) - x_0) \leqq r^2 \quad \text{for} \quad x \in \mathrm{Bd}(S). \tag{1}$$

Then there exists a point $\bar{x} \in S$ such that $f(\bar{x}) = \bar{x}$(4).

Proof. Define for $x \in S$ the function

$$g(x) = \begin{cases} f_1(x) = f(x) & \text{for } x \in X_1 = \{x: \|f(x) - x_0\| \leqq r\} \\ f_2(x) = x_0 + r \dfrac{f(x) - x_0}{\|f(x) - x_0\|} & \text{for } x \in X_2 = \{x: \|f(x) - x_0\| \geqq r\} \end{cases}$$

Then, by the continuity of f, the sets X_1 and X_2 are closed subsets of S, and for $x \in X_1 \cap X_2 = \{x: \|f(x) - x_0\| = r\}$ obviously $f_1(x) = f_2(x)$. Hence, by Lemma 1, $g(x)$ is a continuous function on S. Moreover, since $S = X_1 \cup X_2$ and f is completely continuous on S, it follows that g is also completely continuous on S. Thus by $\| g(x) - x_0 \| \leqq r$ and by the theorem of Schauder [10, Theorem 2] there exists a point $\bar{x}$ such that $\bar{x} = g(\bar{x})$. Supposing $g(\bar{x}) = f_2(\bar{x}) = \bar{x}$, we get

$$r \frac{f(\bar{x}) - x_0}{\|f(\bar{x}) - x_0\|} = \bar{x} - x_0. \tag{2}$$

Hence $\| \bar{x} - x_0 \| = r$, i.e. $\bar{x} \in \mathrm{Bd}(S)$. On multiplying both sides of (2) scalarly by $f(\bar{x}) - x_0$, it follows from (1) that $r \|f(\bar{x}) - x_0\| \leqq r^2$. But then, by the definition of $g(x)$, we have $\bar{x} = g(\bar{x}) = f(\bar{x})$.

LEMMA 2. *Let $f: S \to X$ be a mapping of a sphere $S = \{x: \| x - x_0 \| \leqq r\}$ in the Hilbert space X into X, such that for some $\beta \neq 0$, the mapping $x + \beta f(x)$ is completely continuous on S. Suppose further that for some given $\bar{y} \in X$ we have*

$$(x - x_0, \beta[f(x) - \bar{y}]) \leqq 0 \text{ for every } x \in \mathrm{Bd}(S). \tag{3}$$

Then there exists a point $\bar{x} \in S$, such that $f(\bar{x}) = \bar{y}$.

Proof. We show that Theorem 1 applies to the mapping $h(x) = x + \beta[f(x) - \bar{y}]$. In fact, since $x + \beta f(x)$ is completely continuous, it follows that $h(x)$ is completely continuous, and it remains to verify that (1) holds with f replaced by h. Indeed, for $x \in \mathrm{Bd}(S)$ we have $(x - x_0, x - x_0 + \beta[f(x) - \bar{y}]) = \|x - x_0\|^2 + (x - x_0, \beta[f(x) - \bar{y}]) = r^2 + (x - x_0, \beta[f(x) - \bar{y}])$. Hence by (3), the assumption (1) holds with f replaced by h. By Theorem 1 there exists a point $\bar{x} = h(\bar{x})$, and since $\beta \neq 0$ it follows that $f(\bar{x}) = \bar{y}$.

Putting, in Lemma 2, $x_0 = 0$ and either $\beta = 1$ or $\beta = -1$ we obtain the following

(4) For mappings of finite dimensional spaces, Theorem 1 can be derived from a result of A. Abian and A. B. Brown [1].

THEOREM 2. *If $f: S \to X$ is a mapping of the sphere $S = \{x: \|x\| \leqq r\}$ in the Hilbert space X into X such that for some given $\bar{y}$ either*

(a) *$(x, f(x)) \leqq (x, \bar{y})$ for $\|x\| = r$ and $x + f(x)$ is completely continuous in S or*

(b) *$(x, f(x)) \geqq (x, \bar{y})$ for $\|x\| = r$ and $x - f(x)$ is completely continuous in S, then there exists a point $\bar{x} \in S$, such that $f(\bar{x}) = \bar{y}$.*

The following two examples illustrate the use of Theorem 2.

EXAMPLE 1. Let X be the Euclidean $2n$-dimensional space, $\{a_j\}_{j=1,2\cdots n}$ a sequence of n real numbers, k — a positive integer and consider the mapping $f: X \to X$ defined by

$$(4) \qquad \begin{cases} y_{2j-1} = -x_{2j-1}^{2k-1} - a_j x_{2j} \\ y_{2j} \;\;\;\; = -x_{2j}^{2k-1} + a_j x_{2j-1} \end{cases} \qquad (j = 1, 2, \cdots, n)$$

where $x = (x_1, x_2, \cdots, x_{2n})$ and $y = (y_1, y_2, \cdots, y_{2n})$ are points of X.

Then the image of the unit sphere $S = \{x: \|x\| \leqq 1\}$ contains a sphere $S' = \{y: \|y\| \leqq r_0\}$ with radius $r_0 \geqq (1/2n)^{k-1}$.

Proof. We have $\phi(x) = (x, f(x)) = -\sum_{j=1}^{2n} x_j^{2k}$ and for $\|x\| = 1$ (i.e. for $x \in \mathrm{Bd}(S)$), $\phi(x)$ has a maximum for $x_1 = x_2 = \cdots = x_{2n}$. This maximum equals $\max_{\|x\|=1} \phi(x) = -(1/2n)^{k-1} = -r_0$. Now, if $\|\bar{y}\| \leqq r_0$ then for x with $\|x\| = 1$ we have $|(x, \bar{y})| \leqq \|\bar{y}\| \leqq r_0$ and therefore $(x, \bar{y}) \geqq -r_0$. It follows, that for every $\|\bar{y}\| \leqq r_0$ we have $(x, f(x)) \leqq (x, \bar{y})$ for x with $\|x\| = 1$. Moreover, since X is finite-dimensional, $x + f(x)$ is completely continuous and therefore the assumption (a) of Theorem 2 is satisfied. Thus, by Theorem 2, there exists a point $\bar{x} \in S$ such that $f(\bar{x}) = \bar{y}$. Hence each point $\bar{y} \in S'$ is an image of some point $\bar{x} \in S$.

REMARK 2. Let us note that the result obtained in the above example is in general the best possible. In fact, if X is the real plane (i.e. $n = 1$) and $k = 2$ the mapping (4) has the form

$$(4') \qquad \begin{cases} y_1 = -x_1^3 - a_1 x_2 \\ y_2 = -x_2^3 + a_1 x_1 \end{cases}$$

Therefore, by the result of Example 1, the map of the unit circle under mapping (4′) contains a circle $\|y\| \leqq r_0$ with radius $r_0 \geqq \frac{1}{2}$. Now taking $a_1 = 0$, we obtain $y_1 = -x_1^3$; $y_2 = -x_2^3$, thus $x_1 = -y_1^{1/3}$, $x_2 = -y_2^{1/3}$ and the image of the unit circle $x_1^2 + x_2^2 \leqq 1$ is the set S' of points satisfying $y_1^{2/3} + y_2^{2/3} \leqq 1$. It is easy to see, that the largest circle contained in S' has radius $\frac{1}{2}$.

Similarly it can be shown that for a $(2n-1)$-dimensional Euclidean space X, the map of the unit sphere $\|x\| \leqq 1$ under the mapping:

$$\begin{aligned} y_{2j-1} &= -x_{2j-1}^{2k-1} - a_j x_{2j} \\ y_{2j} &= -x_{2j}^{2k-1} + a_j x_{2j-1}, \qquad (j = 1,2,\cdots,n-1) \\ y_{2n-1} &= -x_{2n-1}^{2k-1} \end{aligned}$$

where $a_j, j = 1,2,\cdots,n-1$, are real constants and k is a positive integer, contains a sphere $S' = \{y : \| y \| \geqq r_0\}$ with $r_0 \geqq 1/(2n-1)^{k-1}$.

EXAMPLE 2. Let $X = L_2[0,1]$ be the Hilbert space of all square integrable functions on the interval $[0,1]$ and consider the mapping $f: S \to X$ of the unit sphere $S = \{x = x(t): \| x \| \leqq 1\}$ into X defined by $f(x) = x + \int_0^1 x^2(u)du$. Then $x - f(x) = - \int_0^1 x^2(u)du$ is a completely continuous mapping on S and for x with $\| x \| = 1$ we have

$$|(x, 1 - \bar{y})| \leqq \| x \| \cdot \| 1 - \bar{y} \| = \| 1 - \bar{y} \|$$

Hence, for x with $\| x \| = 1$ we obtain

$$(x, f(x)) - (x, \bar{y}) = 1 + \int_0^1 x(t)[1 - \bar{y}(t)]dt \geqq 1 - \| 1 - \bar{y} \|.$$

It follows that for $\bar{y} = \bar{y}(t)$ satisfying $1 \geqq \| 1 - \bar{y} \|$ the inequality $(x, f(x)) \geqq (x, \bar{y})$ holds for $x \in \mathrm{Bd}(S)$. Thus the assumption (b) of Theorem 2 is satisfied. By Theorem 2, we obtain that the image $f(S)$ contains a sphere $S' = \{y : \| y - 1 \| \leqq 1\}$.

It is easy to verify that S' is the largest sphere with center $y_0(t) = 1$ contained in the image $f(S)$ of S. In fact, taking any constant function $y(t) = 2 + \varepsilon$ with $\varepsilon > 0$ and assuming that for some $x = x(t)$ there is $f(x) = 2 + \varepsilon$, we obtain $x(t) = \text{const} = c$ and $c^2 + c - (2 + \varepsilon) = 0$. Hence $|c| > 1$ and thus $\| x \| > 1$.

2. In this section some consequences of the contractive-mapping principle and their applications are given. The idea of use of the contractive mapping principle is analogous to that used in [5, p. 148] for finding of the so-called resolvent of a non-linear operator.

Let us call a mapping $g: X \to X$ of a complete metric space X, with metric ρ, into itself γ-contractive $(0 < \gamma < 1)$, if

$$\rho(g(x), g(y)) \leqq \gamma \rho(x, y) \tag{5}$$

holds for all $x, y \in X$. If (5) holds for all x, y belonging to a sphere S in X, then g is called γ-contractive in S.

It is known that

(6) If $g: S \to X$ is γ-contractive in the sphere $S = S(x_0, r)$ contained in a complete metric space X and if for $x_1 = g(x_0)$ we have $\rho(x_0, x_1) \leqq (1 - \gamma)r$, then the sequence $x_n = g(x_{n-1})$, $n = 1,2,\cdots$ converges to the unique solution $x \in S(x_0, r)$ of the equation $g(x) = x$ [7, p. 49, Remark 2].

Let now $f: X \to X$ be a mapping of a Banach space X into itself. We say that the sequence $(\alpha,\beta,\gamma,r,x_0)$ has the property P and write $(\alpha,\beta,\gamma,r,x_0)\in P$ if the mapping $\alpha x-\beta f(x)$ is γ-contractive in the sphere $S=S(x_0,r)$(5).

Note, that if $(\alpha,\beta,\gamma,r,x_0)\in P$ and y is any point of X, then $g(x)=\alpha x-\beta[f(x)-y]$ is γ-contractive in $S(x_0,r)$ and, for $x_1=g(x_0)$ and $y_0=f(x_0)$, we have

$$\rho(x_0,x_1)=\|x_0-\alpha x_0+\beta(y_0-y)\|\leqq|1-\alpha|\cdot\|x_0\|+\beta\|y_0-y\|.$$

Hence, for y satisfying $\|y_0-y\|\leqq[(1-\gamma)r-|1-\alpha|\cdot\|x_0\|]/|\beta|$, we obtain $\rho(x_0,x_1)\leqq(1-\gamma)r$. Therefore it follows from (6) that if $(\alpha,\beta,\gamma,r,x_0)\in P$, then there exists a unique point $x\in S(x_0,r)$, such that $\alpha x-\beta[f(x)-y]=x$, i.e. $y=((1-\alpha)/\beta)x+f(x)$ (if $\beta\neq 0$). In other words:

(7) If $(\alpha,\beta,\gamma,r,x_0)\in P$ and $\beta\neq 0$, then the image $f(S)$ of the sphere $S=S(x_0,r)$ under the mapping $f(x)+((1-\alpha)/\beta)x$ contains a sphere $S(y_0,r_0)$ with center $y_0=f(x_0)$ and radius $r_0=[(1-\gamma)r-|1-\alpha|\cdot\|x_0\|]/|\beta|$.

Noting that if $S_1\subset S_2$ then $f(S_1)\subset f(S_2)$, and putting in (7) $\alpha=1$, we obtain the following

THEOREM 3. *If* $(1,\beta,\gamma,r,x_0)\in P$, $S(x_0,r)\subset S=S(\bar{x},\bar{r})$ *and* $\beta\neq 0$ *then the image* $f(S)$ *of* S *contains a sphere* $S'=S(y_0,r_0)$ *with center* $y_0=f(x_0)$ *and radius* $r_0=((1-\gamma)/|\beta|)r$.

For some applications of this theorem, let us recall the notion of derivative for mappings in Banach spaces.

Let $f:G\to H$ be a mapping of an open set G contained in a Banach space X into a subset H of a Banach space Y. Let $x_0\in G$, and suppose that there exists a linear mapping $A:X\to Y$ such that for every $x\in X$ we have $\lim_{t\to 0}[f(x_0+tx)-f(x_0)]/t=A(x)$. The mapping A is called the derivative of f at the point x_0 and denoted by $f'(x_0)$ or f'.

It can be shown (6) that:

If $[y,x]\subset G$ is an interval and $f:G\to H$ has a derivative at each point of $[y,x]$, then for every linear mapping $U:X\to Y$ we have $\|f(x)-f(y)-U(\Delta y)\|\leqq\sup_{0<\theta<1}\|f'(y+\theta\Delta y)-U\|\cdot\|\Delta y\|$ where $\Delta y=x-y$.

Substituting βf for f and $Ix=x$ for Ux in the last inequality, we obtain that if $f:S\to X$ is a mapping of a sphere $S=S(x_0,r)$ in the Banach space X into X having a derivative f' at every point of S, then for each two points $x,y\in S$ we have $\|x-y-\beta[f(x)-f(y)]\|\leqq\sup_{\xi\in S}\|\beta f'(\xi)-I\|\cdot\|x-y\|$.

Thus, if for some $\beta\neq 0$ and γ, $0<\gamma<1$,

$$\|\beta f'(\xi)-I\|\leqq\gamma \tag{8}$$

is satisfied for every $\xi\in S(x_0,r)$, we obtain that $(1,\beta,\gamma,r,x_0)\in P$.

Therefore, by Theorem 3, we have the following

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(5) An analogous condition was used in [8].

(6) [6, p. 592].

THEOREM 4. *If $f: S \to X$ is a mapping of a sphere $S = S(\bar{x}, \bar{r})$ in a Banach space X into X, having a derivative f' at every point of some sphere $S(x_0, r) \subset S$ and if there exist $\beta \neq 0$ and γ, $0 < \gamma < 1$, such that* (8) *holds, then $f(S)$ contains a sphere $S' = S(y_0, r_0)$ with center $y_0 = f(x_0)$ and radius $r_0 = ((1-\gamma)/|\beta|)r$.*

The next two examples illustrate the use of this theorem in the case $\beta = 1$.

EXAMPLE 3. Let $X = C[0,1]$ be the Banach space of all continuous functions on the interval $[0,1]$, with the norm $\|x\| = \max_{0 \le t \le 1} |x(t)|$. Let $f(x) = x(t) + \int_0^t x^2(u)du$ (7). Then $f'(\xi)x = x(t) + 2 \int_0^t \xi(u)\, x(u)\, du$ and $[I - f'(\xi)]x = 2 \int_0^t \xi(u)x(u)du$. Hence $\|[I - f'(\xi)]x\| \leqq 2 \int_0^1 |\xi(u)| \cdot |x(u)| du$ and $\|I - f'(\xi)\| \leqq 2 \int_0^1 \|\xi(u)\| du = \gamma$.

Taking $\|\xi(u)\| \leqq r$ we thus get, by Theorem 4, $r_0 = (1-2r)r$, the maximum of which is attained for $r = \frac{1}{4}$. Hence the image of the sphere $S(0, \frac{1}{4})$ contains the sphere $S(0, \frac{1}{8})$.

EXAMPLE 4. Let $X = C[0,1]$ and let $f(x) = x(s) + \int_0^1 e^{-st}[x(t) + \frac{1}{2}x^2(t)], dt$, $0 \leqq s \leqq 1$. We have $f'(\xi)x(t) = x(s) + \int_0^1 e^{-st}[1 - \xi(t)]x(t)dt$ and for $\xi(t) \in S(0,1)$ we obtain

$$\|f'(\xi) - I\| = \max_{0 \leqq s \leqq 1} \int_0^1 e^{-st}(1 + \xi(t))dt = \int_0^1 [1 + \xi(t)]\, dt \cdot$$

Let us now look for a fixed sphere $S(x_0, r)$ contained in $S(0,1)$ and find $\sup_{\xi \in S(x_0,r)} \|f'(\xi) - I\|$. For $\xi \in S(x_0, r)$ we have $1 + \xi(t) \leqq 1 + x_0 + r$ and $\int_0^1 [1 + \xi(t)]dt \leqq 1 + r + \int_0^1 x_0(t)dt = \gamma$. Therefore $\|f'(\xi) - I\| \leqq \gamma$ for $\xi \in S(\overset{x}{0}, r)$. By Theorem 4, we obtain $r_0 = (1 - \gamma)r = -r(r + \int_0^1 x_0(t)dt$. Taking $x_0(t) = c = \text{const}$, with $c < 0$, we obtain from $S(x_0, r) \subset S(0,1)$ that $r = 1 + c$ and $r_0 = -r(r + c) = -(1 + c)(1 + 2c)$. This has a maximum for $c = -\frac{3}{4}$. Therefore for $x_0(t) = -\frac{3}{4}$, we have $r = 1 + c = \frac{1}{4}$ and $r_0 = -\frac{1}{4}(-\frac{1}{2}) = \frac{1}{8}$. It follows that the image $f[S(-\frac{3}{4}, \frac{1}{4})]$ of the sphere $S(-\frac{3}{4}, \frac{1}{4})$ contains a sphere with radius $\frac{1}{8}$.

REMARK 3. The results obtained in Examples 3 and 4 are not the best possible. We note also the following consequence of Theorem 1: If $f: S \to X$ is a completely continuous mapping of a sphere $S = S(0,r)$ in a Hilbert space X into X such that $f[\mathrm{Bd}(S)] \subset S$, then there exists a fixed point $x = f(x) \in S$. Indeed, if $f[\mathrm{Bd}(S)] \subset S$ then evidently $(x, f(x)) \leqq r^2$ on $\mathrm{Bd}(S)$, i.e. condition (1) is satisfied (for $x_0 = 0$) and by Theorem 1 there exists a point x such that $x = f(x)$. This result is well known for mappings of spheres in finite-dimensional Euclidean spaces. Another well-known result, which can also be easily derived from Theorem 1, is:

If $f: S \to X$ is a mapping of a sphere in a finite dimensional Euclidean space X into X which is a homeomorphism on $\mathrm{Bd}(S)$ and for which $f(\mathrm{Bd}(S)) = \mathrm{Bd}(S)$, then $S \subset f(S)$.

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(7) This mapping was considered in [3, p. 136].

We shall now prove a theorem, which is an analogue for Banach spaces to a "uniform" version of the theorem on implicit functions for finite dimensional spaces. Before proving this analogue let us make some general remarks.

First of all, we note that in infinite dimensional spaces X an isometric mapping $f: S \to S$ of a sphere $S \subset X$ need not necessarily contain a sphere in X. Indeed, let X be the Hilbert space l_2. i,e., the set of all sequences $(x_1, x_2, \cdots, x_n \cdots,)$ such that $\sum_{i=1}^{\infty} x_i^2 < \infty$, where x_i are real numbers and let S be the unit sphere $S = \{x: \| x \| \leqq 1\}$ in X. Let $f(x) = (0, x_1, x_2, \cdots)$ for $x = (x_1, x_2, \cdots)$. Then $f(S)$ is nowhete dense in X and thus it does not contain any sphere in X.

In what follows, we shall confine ourselves to mappings of the form $x - F(x)$, where $F(x)$ maps a Banach space X into itself.

In the case that $F(x)$ is completely continuous it is known that the image $f(S)$ of a closed unit sphere $S \subset X$ (or even of a closed and bounded set) is a closed subset of X [4, p. 193]. A simple example shows that it is not true that the image $f(S)$ contains a sphere in X. Indeed let us define $F(x) = (x_1, 0, 0, \cdots)$ for $x = (x_1, x_2, \cdots)$ where $x \in l_2$. Then $f(x) = x - F(x) = (0, x_2, x_3, \cdots)$, where $F(x)$ is completely continuous, but $f(S)$ does not contain any sphere in X. Note however, that in this last case we have $\| F'(x) \| = 1$ for every $x \in l_2$. In the case that $\| F' \| \leqq \gamma < 1$ it follows (for $F(x)$ not necessarily completely continuous) that $\| f' - I \| = \| F' \| \leqq \gamma$. Hence, by Theorem 4, the image $f(S)$ of the sphere $S = S(0, r)$ in a Banach space X contains a sphere in X of radius $(1 - \gamma) r$.

THEOREM 5. *If* $\| F'(0) \| \leqq \gamma < 1$, $\| F''(x) \| \leqq K$ *for* $\| x \| \leqq r$ *and* $r \geqq ((1-\gamma)/2K)$, *then the image* $f(S)$ *of the sphere* $S = \{x: \| x \| \leqq r\}$ *contains a sphere of radius* $r_0 = ((1-\gamma)/2)^2 / K$.

Proof. We have $\| F'(x) \| = \| F'(0) + F'(x) - F'(0) \| \leqq \gamma + \| F'(x) - F'(0) \| \leqq \gamma + K \| x \|$ (8). Hence for x satisfying $\| x \| \leqq ((1 - \gamma)/2K)$ we obtain $\| F'(x) \| \leqq (1 + \gamma)/2 < 1$. Therefore $\| f' - I \| = \| F' \| \leqq (1 + \gamma)/2$ and by Theorem 4, the image $f(S)$ of the sphere S contains a sphere of radius $r_0 = ((1-\gamma)/2)^2/K$ and center $y_0 = f(0)$. It is easy to show that f maps the sphere $\{x: \| x \| \leqq ((1-\gamma)/2K\}$ onto $S(y_0, r_0)$ in a one-to-one fashion.

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(8) This last inequality follows from an inequality in [6, p. 592].

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